9.1 (continued)

if $f(x)$ is periodic w/ period 2π and defined on $-\pi < x < \pi$,

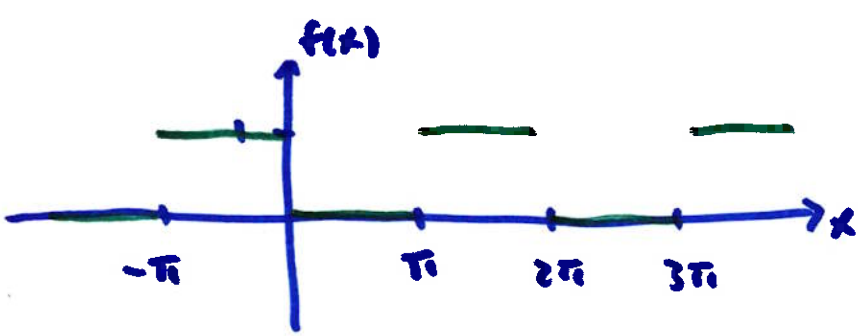
then its Fourier series representation is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n = 0, 1, 2, 3, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad n = 1, 2, 3, \ldots$$

last time: $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } 0 < x < \pi \end{cases}$ period 2π



we found $a_0 = 1$, $a_n = 0$ $n \geq 1$

$$b_n = -\frac{1}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

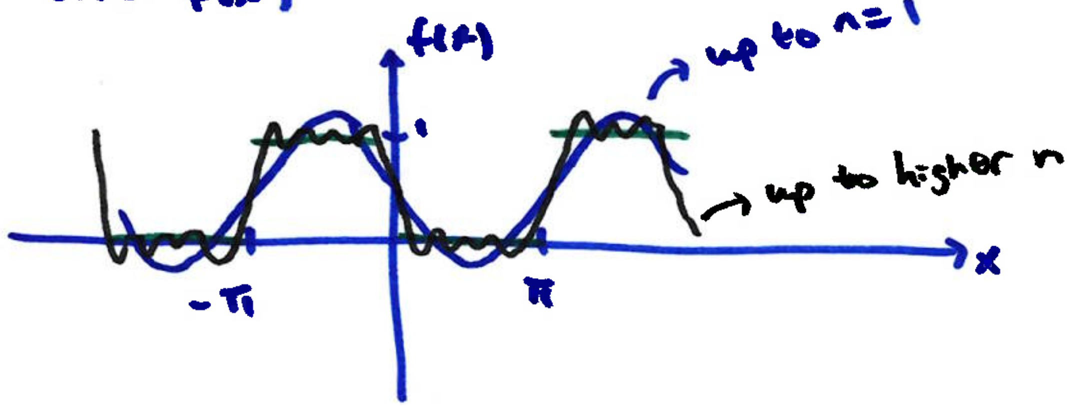
the function's Fourier series representation is

$$f(x) \sim \frac{1}{2}(1) + \sum_{n=1}^{\infty} -\frac{1}{n\pi}\left(1-(-1)^n\right)\sin(nx)$$

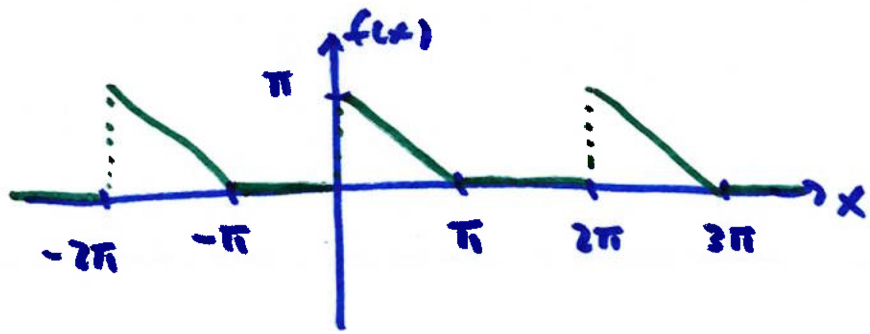
converges to true function $f(x)$

$$\sim \frac{1}{2} - \frac{2}{\pi}\sin(x) - \frac{2}{3\pi}\sin(3x) - \frac{2}{5\pi}\sin(5x) - \dots$$

just like Taylor series, the more terms we include, the closer the Fourier series will resemble the true $f(x)$



another one: $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$ period 2π



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

calculate a_0 separately

$$a_0 = \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} f(x)\,dx}_{\text{area under } f(x)} = \frac{1}{\pi}\left(\frac{1}{2}\cdot\pi\cdot\pi\right) = \frac{1}{2}\pi$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos(nx)\,dx \quad \text{by parts}$$

$$= \ldots$$

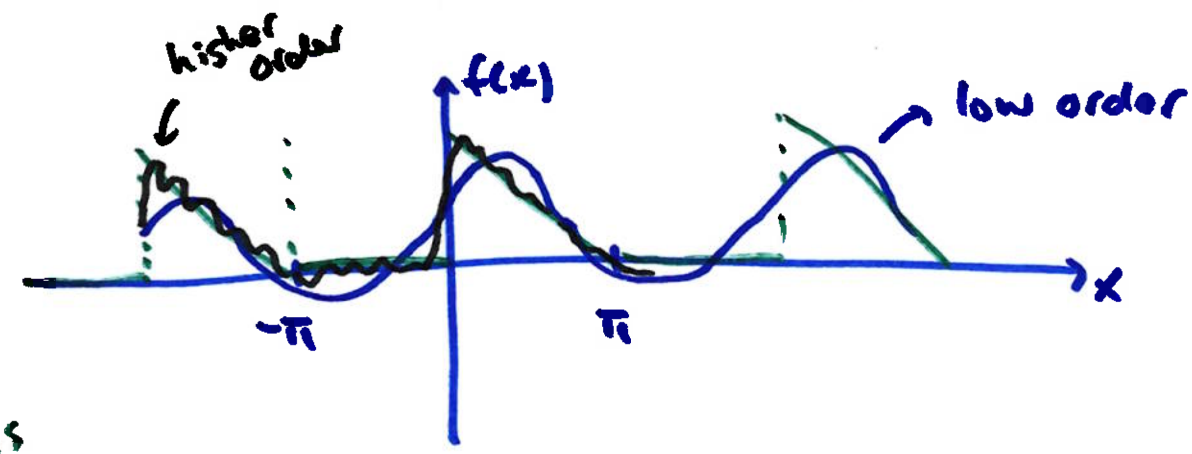
$$= \frac{1}{n^2\pi}\Big(1 - \overbrace{\cos(n\pi)}^{(-1)^n}\Big) = \frac{1}{n^2\pi}\left(1-(-1)^n\right) \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin(nx)\,dx \quad \text{by parts}$$

$$= \cdots = \frac{1}{n} \qquad n = 1, 2, 3, \ldots$$

$$f(x) \sim \frac{1}{2}\left(\frac{\pi}{2}\right) + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(1-(-1)^n)\cos(nx) + \frac{1}{n}\sin(nx)\right]$$

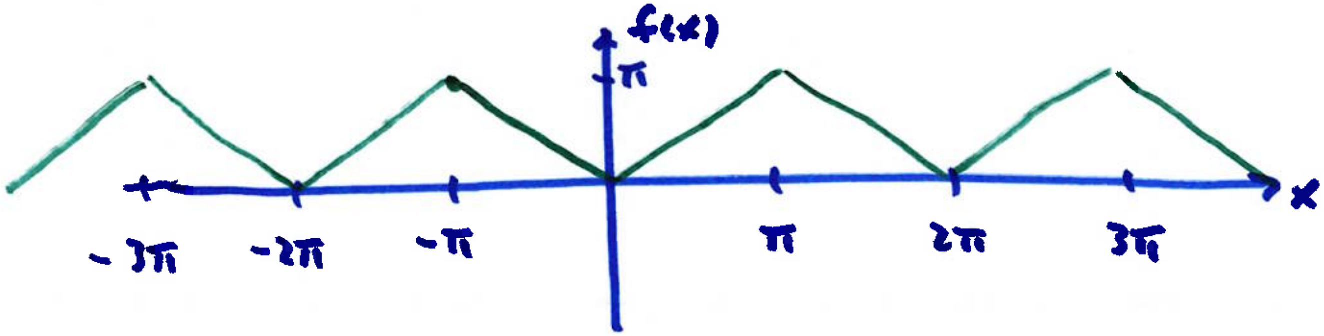
$$\sim \frac{\pi}{4} + \underbrace{\frac{2}{\pi}\cos(x) + \sin(x)}_{n=1} + \underbrace{\frac{1}{2}\sin(2x)}_{n=2} + \underbrace{\frac{2}{9\pi}\cos(3x) + \frac{1}{3}\sin(3x)}_{n=3} + \cdots$$



Fourier series properties discussed further later

notice the overshoot right before and after discontinuities (doesn't go away even when n increases)

also, Fourier series seems to cut through the middle at discontinuities one more example: $f(x) = |x| \quad -\pi < x < \pi \quad$ period 2π



$$f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\frac{1}{2}\cdot 2\pi \cdot \pi\right) = \pi$$

$$a_n = \cdots = \frac{-2}{n\pi}\left(1-(-1)^n\right)$$

$$b_n = \cdots = 0$$